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Magnetic field intermittency and fast dynamo action in random helical flows

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The evolution of passive magnetic fields is considered in random flows made up of single helical waves. In the absence of molecular diffusion the growth rates of all moments of a magnetic field are calculated analytically, and it is found that the field becomes increasingly intermittent with time. The evolution of normal modes of the ensemble-averaged field is determined; it is shown that the flows considered give fast dynamo action, and magnetic field modes with either sign of magnetic helicity may grow.

1. Introduction

Vainshtein & Zel'dovich (1972) posed the question of the existence of kinematic 'fast' dynamos: do there exist flows which generate magnetic field on the convective timescale, in the limit of small non-zero magnetic diffusivity? A related problem is to understand the growth of intermittency that occurs when a magnetic or scalar field is advected in a complex fluid flow. Many recent investigations have focused on deterministic flows, whether steady (see, for example, Arnol'd, *et al.* 1981; Galloway & Frisch 1986; Soward 1987; Gilbert 1988; Falcioni, Paladin & Vulpiani 1989; Gilbert & Childress 1990; Finn, *et al.* 1991) or unsteady (see, for example, Bayly & Childress 1988; Finn & Ott 1988; Ott & Antonsen 1989; Klapper 1992). An alternative approach is to explore magnetic field evolution in random flows. For such flows one can only calculate the evolution of ensemble-averaged quantities, and the detailed stretching and folding processes that occur in a typical realization are lost. However ensemble-averaging annihilates small-scale field components, and this can make diffusion easier to deal with than in deterministic dynamos. In fact when diffusion is weak, it may be neglected entirely when considering the evolution of the ensemble-averaged magnetic field on large scales (Dittrich *et al.* 1984) and if this grows the random flow is a fast dynamo.

An analytical theory of dynamo action in general random flows is lacking (Knobloch 1977). For flows whose correlation time is exactly zero (i.e. delta-correlated flows), dynamo action and intermittency may be analysed in detail (Kazantsev 1967; Kraichnan 1974; Knobloch 1977; Novikov, Ruzmaikin & Sokoloff 1983). Such flows are, however, somewhat unphysical, as a zero correlation time is shorter than the diffusive timescale for magnetic fluctuations of any lengthscale, no matter how small. For a general random flow with a finite correlation time, it is necessary to integrate the motion and stretching of fluid elements in many realizations (Kraichnan 1976*b*; Drummond & Horgan 1986; Drummond & Münch

1990) in order to follow the evolution of magnetic fields. These studies indicate that fast dynamo action occurs (see also Thompson 1990 and Finn *et al.* 1991). In the absence of diffusion, magnetic fields become increasingly intermittent in time: there is uneven stretching and the strongest magnetic fields occupy ever-decreasing volumes of space.

Our aim is to study particular examples of random flows in which magnetic field evolution may be followed analytically without any approximation. Zel'dovich *et al.* (1988) (and references therein) have stressed the value of studying a particular class of homogeneous random flows, which they term 'renewing flows' (or 'renovating flows'). In such a flow time is split into discrete finite intervals of length τ and the random velocity fields in different intervals are independent and identically distributed. Such flows were first introduced by Steenbeck & Krause (1969) and have also been used by Kraichnan (1976*a*) and Vainshtein (1981). Because of the well-defined 'loss of memory' between different time intervals, exact expressions for the evolution of moments of the magnetic field over one time interval may be given.

On general grounds Zel'dovich *et al.* (1988) predict fast dynamo action and intermittency in renewing flows. However, in order to describe magnetic field evolution in specific examples of renewing flows one still needs to know Lagrangian information about the average motion and stretching of fluid elements over the finite time τ, which is difficult to obtain analytically for flows of any complexity. Therefore, in this paper we consider some renewing flows with a simple spatial structure: in each time interval the fluid motion comprises a single random helical wave. For these flows the advection and stretching of fluid elements are easily determined, and this enables us to study dynamo action and the growth of intermittency without recourse to approximation or large numerical simulations. In particular our analysis includes the limit of short correlation time, when τ is much smaller than the turnover time of the waves (Kraichnan 1974; Dittrich *et al.* 1984), as well as the limit of long correlation time (Moffatt 1983).

We begin by introducing renewing flows made up of helical waves (§2). We define the 'isotropic renewing flow', in which the helical waves are isotropically distributed, and the 'ABC renewing flow' in which the wave vectors are always aligned with a coordinate axis, as in the deterministic ABC flows (Dombre *et al.* 1986). In §3 we calculate the growth rates of magnetic field moments, in the case of zero magnetic diffusivity, for the isotropic renewing flow; these characterize the growth of intermittency in the magnetic field. With no magnetic diffusion, magnetic field lines are equivalent to material lines and this allows us to compare our results with the study of line stretching by Drummond & Münch (1990).

In §4 we introduce magnetic diffusion and discuss its effects; in this case it is difficult to follow high-order moments of a magnetic field, and we focus on the ensemble-averaged magnetic field. This field can be decomposed into independent Fourier modes, and weak diffusion has a negligible effect on modes of large scale (Dittrich *et al.* 1984); it is then sufficient to observe growth in some mode at zero diffusion to conclude that the random flow is a fast dynamo. We establish that both the isotropic and ABC renewing flows give fast dynamo action, while the analogous pulsed flows in two-dimensions give decay of ensemble-averaged magnetic fields. Results in the case of the ABC renewing flow are supported by numerical simulations, which indicate that growth of magnetic field occurs in typical realizations, as well as in the ensemble average. We also discuss dynamo action in a renewing flow whose mean flow helicity is zero, but which lacks mirror symmetry. Finally in §5 we offer some concluding remarks.

2. Random renewing flows

We begin by summarizing the properties of single helical waves written in the form

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} \sin(\boldsymbol{q}\cdot\boldsymbol{x}+\psi)+\boldsymbol{b}h\cos(\boldsymbol{q}\cdot\boldsymbol{x}+\psi), \tag{1}$$

together with two conditions

$$\boldsymbol{a}\cdot\boldsymbol{q} = 0, \quad \boldsymbol{b} \equiv \boldsymbol{q}\times\boldsymbol{a}/q, \tag{2}$$

which ensure incompressibility. The wave has wave vector $\boldsymbol{q}$ and phase ψ, while the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ give its polarization. The parameter h (taken to satisfy $|h| \leqslant 1$) controls the relative helicity H of the fluid flow, given by

$$H \equiv \int \boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V \bigg/ \left(\int \boldsymbol{u}^2\,\mathrm{d}V \int \boldsymbol{\omega}^2\,\mathrm{d}V\right)^{\frac{1}{2}} = \frac{2h}{1+h^2}, \tag{3}$$

the integrals being taken over a period of the wave. When $h = \pm 1$, the waves are Beltrami, having maximal relative helicity, $H = \pm 1$. In this case the representation (1) contains some redundancy; a change of the phase of the wave corresponds simply to a rotation of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ about $\boldsymbol{q}$. More formally, for any angle μ the flow (1) is invariant under

$$\boldsymbol{a} \leftarrow \boldsymbol{a}\cos\mu - \boldsymbol{b}h\sin\mu, \quad \boldsymbol{b} \leftarrow \boldsymbol{a}h\sin\mu+\boldsymbol{b}\cos\mu, \quad \psi \leftarrow \psi-\mu, \tag{4}$$

when $h = \pm 1$. The kinetic energy density of the flow is

$$E_{\mathrm{K}} = \tfrac{1}{4}(1+h^2)\,a^2; \tag{5}$$

and it is convenient to introduce the turnover time T of the flow, defined by

$$T^{-1} = q(2E_{\mathrm{K}})^{\frac{1}{2}}. \tag{6}$$

Now suppose that such a wave acts over a finite time interval $(n-1)\,\tau \leqslant t < n\tau$. During this interval a particle is advected from $\boldsymbol{x}$ to $M(\boldsymbol{x})$, where

$$M(\boldsymbol{x}) = \boldsymbol{x}+\boldsymbol{a}\tau\sin(\boldsymbol{q}\cdot\boldsymbol{x}+\psi)+\boldsymbol{b}h\tau\cos(\boldsymbol{q}\cdot\boldsymbol{x}+\psi). \tag{7}$$

In the absence of molecular diffusion a passive magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ is transported according to

$$\boldsymbol{B}(\boldsymbol{x},n\tau) = \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{B}(M^{-1}(\boldsymbol{x}),(n-1)\,\tau), \tag{8}$$

where the Jacobian of the transformation M is given by

$$J_{ij}(\boldsymbol{x}) = \partial x_i/\partial(M^{-1}\boldsymbol{x})_j = \delta_{ij}+a_i q_j \tau\cos(\boldsymbol{q}\cdot\boldsymbol{x}+\psi)-b_i q_j h\tau\sin(\boldsymbol{q}\cdot\boldsymbol{x}+\psi). \tag{9}$$

Equation (8) also governs a line element advected in the flow. The equation for the vector potential $\boldsymbol{A}$ of the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ is

$$\boldsymbol{A}(\boldsymbol{x},n\tau) = (\boldsymbol{J}^{-1}(\boldsymbol{x}))^{\mathrm{T}}\cdot\boldsymbol{A}(M^{-1}(\boldsymbol{x}),(n-1)\,\tau), \tag{10}$$

in a certain gauge (Roberts 1967). This equation also governs a directed area element advected by the flow and the gradient of a passive scalar. Note that in this gauge the magnetic helicity density is constant following a fluid element

$$\boldsymbol{A}(\boldsymbol{x},n\tau)\cdot\boldsymbol{B}(\boldsymbol{x},n\tau) = \boldsymbol{A}(M^{-1}(\boldsymbol{x}),(n-1)\tau)\cdot\boldsymbol{B}(M^{-1}(\boldsymbol{x}),(n-1)\,\tau). \tag{11}$$

The conservation of total magnetic helicity is a consequence of the preservation of the magnetic field line topology in a perfectly conducting fluid (Moffatt 1969).

Now let us construct some random flows from such waves. Molchanov, Ruzmaikin & Sokoloff (1984) define a 'renewing flow' as a random flow which is statistically homogeneous, and which is independent and identically distributed in distinct time intervals $(n-1)\tau \leqslant t < n\tau$, with $n = 1, 2, 3, \ldots$. We consider renewing flows which take the form of a single wave (1) chosen at random in each time interval. The flow is then specified by giving the distribution of the parameters $\boldsymbol{q}, \boldsymbol{a}, \psi, h$ for any time interval.

An isotropic ensemble of flows is obtained by letting the vector $\boldsymbol{q}$ be distributed uniformly on the sphere of radius q, and the phase ψ be distributed uniformly on the interval $(0, 2\pi)$. Once $\boldsymbol{q}$ has been chosen, the vector $\boldsymbol{a}$ is distributed uniformly on the circle of radius a in the plane perpendicular to $\boldsymbol{q}$. The parameters a, q, h, τ are non-random and describe completely the renewing flow, which is clearly isotropic and homogeneous. We call this the isotropic renewing flow.

An interesting anisotropic ensemble is obtained by choosing $\boldsymbol{q}$ out of the set of standard unit basis vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, each basis vector having a probability $\frac{1}{3}$ of being chosen. Again ψ is uniformly distributed on $(0, 2\pi)$ and $\boldsymbol{a}$ is uniformly distributed on the circle of radius a in the plane perpendicular to $\boldsymbol{q}$. Each flow in this ensemble is periodic in the cube $(0, 2\pi)^3$. This ensemble is specifically designed to resemble the steady ABC flows studied by Arnol'd & Korkina (1983) and Galloway & Frisch (1986), and the unsteady flows studied more recently by Finn & Ott (1988) and Otani (1988). We therefore call this the ABC renewing flow.

3. Magnetic field intermittency

Because a renewing flow has a simple temporal structure, with a well-defined loss of memory at times $t = n\tau$, the advection of passive fields may be studied in some detail. We begin by considering the growth of the moments of the magnetic field, in the absence of any molecular diffusion. We use results that rely crucially on the homogeneity of the renewing flow, and it is necessary to suppose that the initial conditions for the magnetic field are also random and homogeneous. We use $\langle \cdot \rangle$ to denote an average over both the ensemble of flows and the initial conditions for the field.

Consider a single-point moment of the field $\langle (B_i B_j \ldots B_l)(\boldsymbol{x}, t)\rangle$, which is by hypothesis homogeneous at $t = 0$. In a renewing flow such a moment remains homogeneous at later times, and setting

$$R_{ij\ldots l}(n\tau) = \langle (B_i B_j \ldots B_l)(\boldsymbol{x}, n\tau)\rangle, \tag{12}$$

we have
$$R_{i'j'\ldots l'}(n\tau) = H_{i'ij'j\ldots l'l} R_{ij\ldots l}((n-1)\tau), \tag{13}$$

where
$$H_{i'ij'j\ldots l'l} = \langle (J_{i'i} J_{j'j} \ldots J_{l'l})(\boldsymbol{x})\rangle \tag{14}$$

(Molchanov *et al.* 1984). The average here is over the distribution of flows in any one time interval.

We illustrate the derivation of (13) in the case of the second moment; from the Cauchy solution (8):

$$\langle B_i(\boldsymbol{x}, n\tau) B_k(\boldsymbol{x}, n\tau)\rangle = \langle J_{ij}(\boldsymbol{x}) J_{kl}(\boldsymbol{x}) B_j(M^{-1}(\boldsymbol{x}), (n-1)\tau) B_l(M^{-1}(\boldsymbol{x}), (n-1)\tau)\rangle. \tag{15}$$

Averaging over the distribution of flows in the period $(0,(n-1)\tau)$ and assuming inductively that the second moment is homogeneous at time $t=(n-1)\tau$ gives

$$\langle B_i(\boldsymbol{x},n\tau)\,B_k(\boldsymbol{x},n\tau)\rangle = \langle J_{ij}(\boldsymbol{x})\,J_{kl}(\boldsymbol{x})\rangle R_{jl}((n-1)\,\tau). \tag{16}$$

Thus the second moment is homogeneous at time $n\tau$ and given by (13), (14).

We can apply this result to the evolution of a magnetic field carried by the isotropic renewing flow. For the second moment substitute two copies of the Jacobian (9) into (14) and average over the phase ψ to obtain

$$H_{ijkl} = \langle \delta_{ij}\,\delta_{kl} + \tfrac{1}{2}q_j\,q_l\,(a_i\,a_k + b_i\,b_k\,h^2)\,\tau^2\rangle. \tag{17}$$

Averaging over the directions of $\boldsymbol{a}$ and then $\boldsymbol{q}$ finally yields

$$H_{ijkl} = \delta_{ij}\,\delta_{kl} + q^2\tau^2 E_{\mathrm{K}}\,(\tfrac{1}{3}\delta_{ik}\,\delta_{jl} - \tfrac{1}{5}\delta_{(ij}\,\delta_{kl)}), \tag{18}$$

where the subscripted round brackets denote symmetrization, an average over permutations of the indices contained therein; for example

$$\delta_{(ij}\,\delta_{kl)} = \tfrac{1}{3}(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}).$$

Suppose now that the initial magnetic field is isotropic as well as homogeneous, so that

$$R_{ij}(t) = 2E_{\mathrm{M}}(t)\,\delta_{ij}, \tag{19}$$

$E_{\mathrm{M}}(t)$ being the magnetic energy density. Then from (18):

$$E_M(n\tau) = (1 + \tfrac{2}{3}q^2\tau^2 E_{\mathrm{K}})^n E_{\mathrm{M}}(0). \tag{20}$$

The magnetic energy grows exponentially because of the stretching of field lines, as we would expect in the absence of molecular diffusion.

To examine possible intermittency in the growing magnetic field we need to find higher moments. Although they may be found from (13), (14) their calculation can be simplified using certain results proved by Kraichnan (1974), Zel'dovich *et al.* (1984) and Drummond & Münch (1990). We give a derivation suited to the development so far. We now assume that the renewing flow and the initial magnetic field are isotropic, as well as homogeneous.

Define moments of the magnetic field amplitude by

$$\overline{B^p}(t) = \langle |\boldsymbol{B}(\boldsymbol{x},t)|^p\rangle; \tag{21}$$

for even values of p these quantities determine the single-point pth-order moments (12), which take the isotropic form:

$$R_{ij\ldots kl}(n\tau) = \overline{B^p}(n\tau)\,c_p\,\delta_{(ij}\ldots\delta_{kl)}, \tag{22}$$

where

$$c_p^{-1} \equiv \delta_{ij}\ldots\delta_{kl}\,\delta_{(ij}\ldots\delta_{kl)}. \tag{23}$$

To follow the growth of these moments, contract (13) with $\delta_{i'j'}\ldots\delta_{k'l'}$ and use (14) to obtain

$$\overline{B^p}(n\tau)/\overline{B^p}((n-1)\,\tau) = c_p\langle J_{i'i}\,J_{j'j}\ldots J_{k'k}\,J_{l'l}\rangle\delta_{i'j'}\ldots\delta_{k'l'}\delta_{(ij}\ldots\delta_{kl)} \tag{24}$$

$$= c_p\langle(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J})_{ij}\ldots(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J})_{kl}\rangle\delta_{(ij}\ldots\delta_{kl)} \tag{25}$$

(omitting the argument of the Jacobian matrices). The index contractions may be written more compactly by introducing a vector variable $\boldsymbol{X}$ and the Laplacian operator $\Delta_{\boldsymbol{X}} = \partial^2/\partial X_i\,\partial X_i$:

$$\overline{B^p}(n\tau)/\overline{B^p}((n-1)\tau) = (\Delta_X^{p/2}\langle(\boldsymbol{X}\cdot\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}\cdot\boldsymbol{X})^{p/2}\rangle)/(\Delta_X^{p/2}X^p), \tag{26}$$

(with $X = |\boldsymbol{X}|$). Now the averaged term in this equation is independent of the direction of the vector $\boldsymbol{X}$ because the renewing flow is isotropic and so the equation finally reduces to

$$\overline{B^p}(n\tau)/\overline{B^p}((n-1)\tau) = \langle(\hat{\boldsymbol{X}}\cdot\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}\cdot\hat{\boldsymbol{X}})^{p/2}\rangle, \tag{27}$$

where $\hat{\boldsymbol{X}} = \boldsymbol{X}/X$ is any unit vector. This result was obtained by Molchanov *et al.* (1984) for any real value of p, by considering the stretching of individual vectors.

Moving the symmetrization operation in line (24) from the unprimed to the primed indices gives the equivalent form:

$$\overline{B^p}(n\tau)/\overline{B^p}((n-1)\tau) = \langle\hat{\boldsymbol{X}}\cdot\boldsymbol{J}\cdot\boldsymbol{J}^{\mathrm{T}}\cdot\hat{\boldsymbol{X}})^{p/2}\rangle. \tag{28}$$

The equivalence of (27), (28) for the moments of the field has been discussed by Kraichnan (1974) and Drummond & Münch (1990). These authors note that as a consequence the moments of a field $\boldsymbol{A}(\boldsymbol{x},t)$ that evolves according to (10) are governed by the time-reversed ensemble of random flows (see also Bayly & Childress 1989). The particular renewing flows we consider are time-reversible, and so the equations governing the evolution of the statistical properties for $\boldsymbol{A}(\boldsymbol{x},t)$ are the same as those for $\boldsymbol{B}(\boldsymbol{x},t)$. Thus in these flows, any growth of intermittency is the same for magnetic fields, vector potentials and scalar gradients, as well as line and area elements. From (**11**) all moments of the magnetic helicity density are conserved in time, in the gauge for which (10) holds.

For the isotropic renewing flow defined in §2:

$$\overline{B^p}(n\tau)/\overline{B^p}((n-1)\tau) = \langle L^p\rangle, \tag{29}$$

where we may take either

$$L^2 = 1 + aq\tau\sin 2\theta\,(\cos\psi'\cos\phi - h\sin\psi'\sin\phi) + (aq\tau)^2\cos^2\theta\,(\cos^2\psi' + h^2\sin^2\psi'), \tag{30}$$

from (27), or

$$L^2 = 1 + aq\tau\sin 2\theta\,(\cos\psi'\cos\phi - h\sin\psi'\sin\phi) + (aq\tau)^2\sin^2\theta\,(\cos\psi'\cos\phi - h\sin\psi'\sin\phi)^2, \tag{31}$$

from (28), this giving a useful check on numerical calculations. Here $\psi' \equiv (\boldsymbol{q}\cdot\boldsymbol{x}+\psi)$ and is distributed uniformly. θ is the angle between $\boldsymbol{q}$ and $\hat{\boldsymbol{X}}$, so that $\cos\theta$ is uniform on the interval $(-1,1)$. ϕ is the angle between $\boldsymbol{a}$ and the projection of $\hat{\boldsymbol{X}}$ perpendicular to $\boldsymbol{q}$ and is distributed uniformly. For comparison with the results of Drummond & Münch (1990) and Molchanov *et al.* (1984) we introduce stretching exponents γ_p defined by

$$\left.\begin{aligned} \gamma_p &= (p\tau)^{-1}\log\langle L^p\rangle. \quad p \neq 0, \\ \gamma_0 &= \lim_{p\to 0}\gamma_p = \tau^{-1}\langle\log L\rangle. \end{aligned}\right\} \tag{32}$$

Thus the exponential rate of increase of the pth moment is $p\gamma_p$ over long periods of time (for $p \neq 0$).

For small values of $aq\tau \ll 1$ and $p = O(1)$, (32) may be expanded (Drummond & Münch 1990), and the averages performed to give

$$\gamma_p = (\tau/T^2)\,(\tfrac{1}{6} + (\tfrac{1}{2}p - 1)\tfrac{1}{15}) + O(\tau^3/T^4). \tag{33}$$

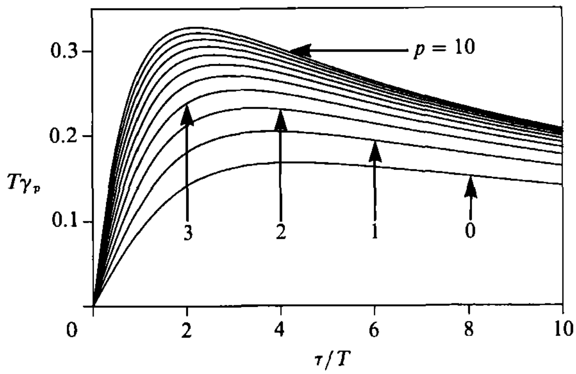

FIGURE 1. Rates of growth of magnetic field moments in the isotropic renewing flow without magnetic diffusion. $T\gamma_p$ is plotted against τ/T for $p = 0, 1, 2, \ldots, 10$ and $|h| = 1$.

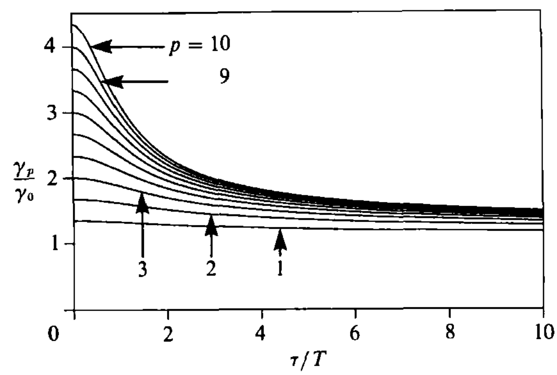

FIGURE. 2. Ratios γ_p/γ_0 as functions of τ/T for $p = 1, 2, \ldots,$ 10 in the isotropic renewing flow with $|h| = 1$.

This linear dependence of the stretching exponent on the index p was obtained by Kraichnan (1974) in the limit of a delta-correlated flow and by Drummond & Münch (1990) in an expansion taking into account moments of up to second order in the velocity field.

For any values of the parameters, the stretching exponents (32) may be evaluated numerically. We begin by taking $h = \pm 1$, so that the average over ϕ is redundant. We measure the correlation time τ and the growth rates γ_p relative to the turnover time which is simply $T = (aq)^{-1}$. Figure 1 shows $T\gamma_p$ against τ/T for $p = 0, 1, 2, \ldots,$ 10. For any value of $\tau/T, T\gamma_p$ increases as a function of p. This means that the distribution of magnetic field becomes increasingly intermittent with time, the strongest values of the field being concentrated in ever smaller regions of space. Such intermittency was predicted by Zel'dovich *et al.* (1984) on general grounds, and has been observed in deterministic chaotic flows (Ott & Antonsen 1989; Falcioni *et al.* 1989) as well as random flows (Kraichnan 1976*b*; Drummond & Münch 1990). Note that the growth rate of energy, $2\gamma_2$, is more than twice that of the mean field strength, γ_1; this behaviour is typical when there is zero magnetic diffusion (Hoyng 1987*a, b*).

Now consider varying τ/T; in the short correlation time limit $\tau/T \to 0$, the growth in moments is weak and given approximately by (33). As τ/T is increased, the growth

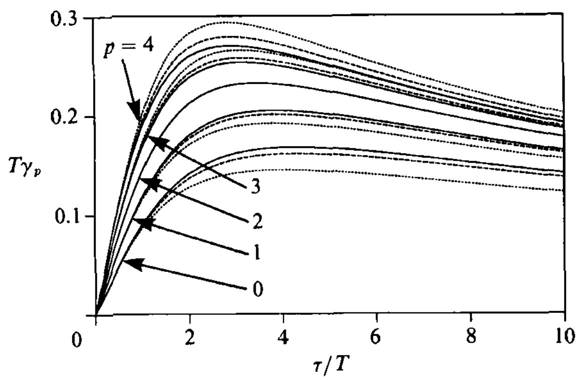

FIGURE 3. Rates of growth of magnetic field moments for varying helicities in the isotropic renewing flow. Curves of $T\gamma_p$ against τ/T are plotted for $p = 0, 1, 2, 3, 4$ with $h = 0$ (......), $\frac{1}{2}$ (------) and 1 (—). Note that the curves for different values of h coincide when $p = 2$.

rates peak and then decay slowly. For example the maximum growth in magnetic energy occurs when $\tau \approx 3/aq$. For a direct comparison with the results of Drummond & Münch (1990), we plot γ_p/γ_0 for different values of τ/T (figure 2). In the short correlation time limit, γ_p/γ_0 tends to $1+\frac{1}{3}p$ given by (33). For increasing values of τ/T, γ_p/γ_0 decreases, particularly strongly for large values of p. Drummond & Münch (1990) calculate γ_p/γ_0 for $p = 1, 2, 3$ and $\tau/T = 1, 2, 3$ and also find values smaller than those given by the short correlation time limit. Unlike us, they observe a weak increase with τ/T. Presumably this reflects the complex structure of their flow fields, which are likely to stretch magnetic field more rapidly than ours when τ/T is large.

Finally we examine the effects of varying h, the helicity of the renewing ensemble. Figure 3 shows $T\gamma_p$ for $p = 0, 1, 2, 3, 4$ and $h = 0, \frac{1}{2}, 1$. The rate of growth of energy, $2\gamma_2$, is independent of helicity from (20) and there is only a weak variation with h for $p \neq 2$, as observed by Drummond & Münch (1990).

4. Dynamo action in renewing flows

We have seen how a magnetic field is stretched by a random renewing flow; the magnetic energy increases while the field becomes ever more intermittent. However, there is presumably also a continual refinement of the scale of the field, and in a realistic situation magnetic diffusion must eventually become important. In this section we study the growth of magnetic field in the presence of weak molecular diffusion. In this case following moments of high order becomes impractical, since diffusion couples nearby points; in order to calculate the growth in the single-point pth moment, one has to follow the evolution of the whole p-point pth moment. However, the first moment, which gives the mean (ensemble-averaged) field, is tractable (Dittrich *et al.* 1984), and the second moment has been studied by Kazantsev (1967) and Novikov *et al.* (1983) when the velocity field is delta-correlated in time and mirror-symmetric. We study the growth of the first moment in the isotropic and ABC renewing flows.

When the fluid is imperfectly conducting, the magnetic field evolution is no longer given by the Cauchy solution (8) but is governed by

$$\boldsymbol{B}(\boldsymbol{x}, n\tau) = \int \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{B}(\boldsymbol{y}, (n-1)\tau)\, \mathrm{d}^3 \boldsymbol{y}, \tag{34}$$

where the Green's function $\boldsymbol{G}$ is random, depending on which random wave (1) is acting during the interval $((n-1)\tau, n\tau)$. The homogeneity of a renewing flow allows us to decompose the mean magnetic field into Fourier modes:

$$\langle \boldsymbol{B}(\boldsymbol{x},t)\rangle \equiv \int \tilde{\boldsymbol{B}}(\boldsymbol{k},t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{k}, \tag{35}$$

with $\boldsymbol{k}\cdot\tilde{\boldsymbol{B}}(\boldsymbol{k},t)=0$. Note that in this section we do not require that the initial magnetic field be isotropic or homogeneous. These magnetic field modes evolve by:

$$\tilde{\boldsymbol{B}}(\boldsymbol{k},n\tau) = \tilde{\boldsymbol{G}}(\boldsymbol{k})\cdot\tilde{\boldsymbol{B}}(\boldsymbol{k},(n-1)\tau), \tag{36}$$

where the 'response tensor' $\tilde{\boldsymbol{G}}(\boldsymbol{k})$ is given by

$$\tilde{\boldsymbol{G}}(\boldsymbol{k}) = \left\langle \int \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y})\exp(\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x}))\mathrm{d}^3\boldsymbol{x}\right\rangle \tag{37}$$

(see e.g. Kraichnan 1976*a*). Equivalent equations have also been obtained by Dittrich *et al.* (1984), who represent diffusion by taking an average of the Cauchy solution over random paths. Once the matrix $\tilde{\boldsymbol{G}}(\boldsymbol{k})$ has been computed, the evolution of a Fourier mode is given by (36). Exponential growth occurs when the field vector is an eigenvector of $\tilde{\boldsymbol{G}}(\boldsymbol{k})$ belonging to an eigenvalue σ of magnitude greater than unity, the growth rate being $\lambda = \tau^{-1}\log|\sigma|$. Growth of any mode implies growth of the magnetic energy; however, the growth rate of energy will generally be greater than that of the most unstable mode because of phase mixing between different members of the ensemble (Hoyng 1987*a*, *b*).

In the limit of weak diffusion, the growth rate of a given magnetic field mode $\tilde{\boldsymbol{B}}(\boldsymbol{k},n\tau)$ tends to the growth rate for zero diffusion (Dittrich *et al.* 1984); this greatly simplifies the fast dynamo problem for random flows. We illustrate this in a simple model in which we 'pulse' helical waves (Bayly & Childress 1988). Split each time interval of length τ into two equal sub-intervals. In the first sub-interval the field undergoes frozen field advection according to (8), the velocity field being twice that in (1), and in the second sub-interval the fluid is stationary and the fields diffuse with a diffusion coefficient of 2η. The Green's function for this pulsed renewing flow is

$$G_{ij}(\boldsymbol{x},\boldsymbol{y}) = (4\pi\eta\tau)^{-\frac{3}{2}}\exp(-(\boldsymbol{x}-M\boldsymbol{y})^2/4\eta\tau)\,J_{ij}(M\boldsymbol{y}), \tag{38}$$

and the response tensor is given by

$$\tilde{\boldsymbol{G}}(\boldsymbol{k}) = \langle \boldsymbol{J}(\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{k}\cdot(M^{-1}\boldsymbol{x}-\boldsymbol{x}))\rangle\exp(-\eta\tau k^2). \tag{39}$$

From (39), we see that for a magnetic mode of fixed wavenumber k, the effect of diffusion is uniformly small in the fast dynamo limit $\eta\rightarrow 0$. This is because the small-scale fields generated by advection in each realization of the flow average to zero on taking the ensemble average (Kraichnan 1976*a*). Thus the effect of weak diffusion in a renewing flow is simply to provide a cutoff at high wavenumbers of any dynamo action, and so to check for fast dynamo action it is legitimate to set $\eta=0$.

We consider in turn the isotropic and ABC renewing flows. The calculation is simplest for the isotropic renewing flow, whose spherical symmetry makes the results straightforward to interpret. The ABC renewing flow is less symmetric than the isotropic flow, but it has the attractive property that it can be thought of as acting on a bounded domain, namely 2π-periodic space. When the parameter h is zero in these examples, the magnetic field does not grow in time. However, if we let h be a

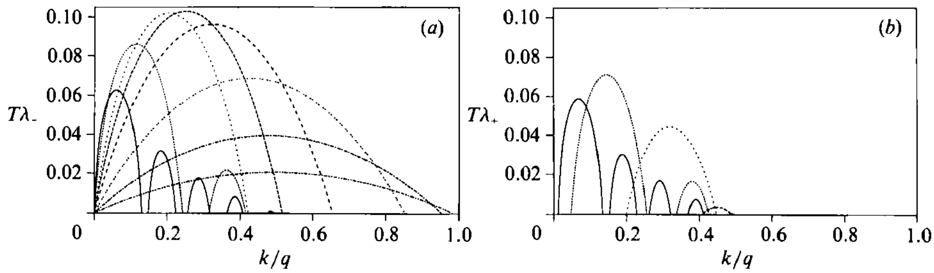

FIGURE 4. Graphs of the growth rates $T\lambda_{\pm}$ of magnetic field modes with positive and negative helicity against k/q for the isotropic renewing flow with $h = 1$ and $\tau/T = 32$ (—), 16 (······), 8 (·-·-·), 6 (----), 4 (-·-·-), 2 (·-·-·-·), 1 (-·-·-) and $\frac{1}{2}$ (·-·-·-·-). In (*a*) and (*b*), $T\lambda_-$ and $T\lambda_+$ are plotted respectively, using the same scales.

random variable instead of a constant parameter, it is possible to have mean field growth in renewing flows with zero mean helicity. We illustrate this using a renewing flow with a simple asymmetric helicity distribution.

4.1. *The isotropic renewing flow*

The response tensor (39) is most easily evaluated when the relative helicity is maximal ($h = \pm 1$) and the wavevectors $\boldsymbol{q}$ are distributed isotropically over the sphere of radius q. Then, for zero diffusion,

$$\tilde{G}_{ij}(\boldsymbol{k}) = g_0(ak\tau)\,\delta_{ij} + g_1(ak\tau)\,(\mathrm{i}aqh\tau/2k)\,\epsilon_{ijk}\,k_k, \tag{40}$$

where $g_0(s) = (\sin s)/s$ and $g_1(s) = (\sin s)/s^2 - (\cos s)/s$.

One eigenvector of $\tilde{\boldsymbol{G}}(\boldsymbol{k})$ is always $\boldsymbol{k}$, corresponding to the eigenvalue $g_0(ak\tau)$. Since magnetic modes must be divergence-free, this eigenvector has no relevance to the dynamo problem. The relevant eigenvectors lie in the plane orthogonal to $\boldsymbol{k}$. If we choose a coordinate system with the z-axis pointing along $\boldsymbol{k}$, the eigenvectors of $\tilde{\boldsymbol{G}}(\boldsymbol{k})$ perpendicular to $\boldsymbol{k}$ take the form $\tilde{\boldsymbol{B}}(\boldsymbol{k}) = (\mp\mathrm{i}, 1, 0)$ with corresponding eigenvalues:

$$\sigma_{\pm} = g_0(ak\tau) \mp (\tfrac{1}{2}aqh\tau)g_1(ak\tau), \tag{41}$$

and growth rates $\lambda_{\pm}$. The eigenvectors are helical waves of the ensemble-averaged field, with relative helicity ± 1.† These results are exact for this model. Equations (40) and (41) may be approximated in the limit $\tau/T \ll 1$ of short correlation time and $k/q \leqslant O(1)$ to recover results of mean-field electrodynamics (Dittrich *et al.* 1984).

We non-dimensionalize all quantities using the turnover time T and the wavenumber q of the flow. Figure 4(*a*) shows the growth rate $T\lambda_-$ of magnetic field modes with negative helicity as a function of scale k/q, for different values of the correlation time τ/T and for $h = 1$. These modes are always unstable at large scales $k/q \ll 1$. Thus these renewing flows give fast dynamo action. For small values of the correlation time there is just one range of scales, or window, of dynamo action. However, as τ/T is increased, the action of the helical waves becomes more persistent, and several windows of unstable magnetic modes appear. Figure 4(*b*) shows the growth rates $T\lambda_+$ of magnetic modes with positive helicity. For small values of the correlation time there is no growth. However, as the persistence of the flow is increased, a window of dynamo action appears for $t/T \approx 6$, and then several windows appear. Thus sufficiently persistent random helical waves can amplify

† The helicity meant here is that of the mean field, $\langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle$, rather the ensemble-averaged magnetic helicity, $\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle$, which is not readily calculated.

magnetic field modes of both signs of helicity. The growth can even occur for modes of the same spatial scale.

For a general value of h, the magnetic response function is still given by (40), but now

$$g_0(s,h) = \langle \sin(s\chi)/(s\chi) \rangle,$$
$$g_1(s,h) = \langle \chi^{-1}[\sin(s\chi)/(s\chi)^2 - \cos(s\chi)/(s\chi)] \rangle,$$

where $\chi^2 = \cos^2\phi + h^2 \sin^2\phi$, and ϕ is distributed uniformly in $(0, 2\pi)$. In fact, ϕ is the angle between $\boldsymbol{a}$ and the projection of $\boldsymbol{k}$ perpendicular to $\boldsymbol{q}$ in each realization of the wave (1). As $|h|$ is decreased from 1, dynamo action becomes less effective; at $h = 0$ all magnetic field modes decay at the same rate as scalar modes of the same scale.

Dynamo action is impossible in two-dimensional flows (Cowling 1934; Zel'dovich 1957); in the context of random waves all magnetic field modes decay to zero. The waves analogous to those of equation (1) are of the form $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a}\sin(\boldsymbol{q}\cdot\boldsymbol{x}+\psi)$ with the distributions of $\boldsymbol{q}$ and $\boldsymbol{a}$ restricted to some specific plane. Again $\boldsymbol{a}$ is required to be perpendicular to $\boldsymbol{q}$ and ψ is uniformly distributed in $(0, 2\pi)$ to ensure homogeneity of the ensemble. In this case

$$\tilde{G}_{ij}(\boldsymbol{k}) = \delta_{ij}\langle \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}\tau\sin(\boldsymbol{q}\cdot\boldsymbol{x}+\psi))\rangle$$

and so all modes decay, as the modulus of the term $\langle\cdot\rangle$ above is less than unity. In two dimensions, the second and higher moments must also decay when there is diffusion; this does not contradict the results of §3, which are only applicable for zero diffusion.

4.2. *The ABC renewing flow*

The only difference between the isotropic and ABC renewing flows, as far as calculations are concerned, is the set of $\boldsymbol{q}$-vectors over which the average is taken. The important difference is that the ABC renewing flow can be thought of as an ensemble of flows on a *compact* domain while the isotropic renewing flow cannot. The fact that the present random dynamo mechanism works on compact domains is non-trivial, as there are other random models (e.g. Zel'dovich *et al.* 1984) that work on unbounded domains but not bounded ones.

The response tensor for the ABC renewing flow is obtained by substituting the unit coordinate vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ for q in (7), (9) and averaging (39). The general formula is a sum of six Bessel functions, each with a different argument.

For particular configurations of the magnetic field mode, we can obtain simpler formulae. If the relative helicity is maximal, $h = \pm 1$, and the wavevector $\boldsymbol{k}$ is aligned with one of the coordinate axes, say the z-axis, then

$$\tilde{G}_{ij}(\boldsymbol{k}) = \delta_{ij}[\tfrac{1}{3}+\tfrac{2}{3}J_0(ak\tau)] + \tfrac{1}{3}(\mathrm{i}ah\tau)\,\epsilon_{ilm}\frac{k_m}{k}(\delta_{l1}\delta_{j1}+\delta_{l2}\delta_{j2})\,J_1(ak\tau). \tag{42}$$

The eigenvectors relevant for dynamo action are again in the plane orthogonal to $\boldsymbol{k}$. In fact, they are the same as the eigenvectors for the isotropic flow, except now the corresponding eigenvalues are

$$\sigma_{\pm} = \tfrac{1}{3}+\tfrac{2}{3}J_0(ak\tau) \mp \tfrac{1}{3}ah\tau\, J_1(ak\tau). \tag{43}$$

In figure 5 we plot $T\lambda_{\pm}$ as functions of k for various values of τ/T. The results are qualitatively the same as in the isotropic ensemble. In particular, for small values of τ/T dynamo action occurs only for magnetic modes with negative helicity, while for large τ/T magnetic modes of either sign of helicity can grow.

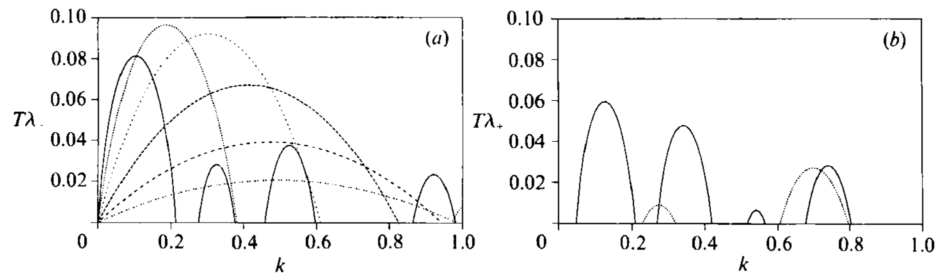

FIGURE 5. Graphs of the growth rates $T\lambda_{\pm}$ of magnetic field modes against k for the ABC renewing flow with $h = 1$ and $\tau/T = 16$ (—), 8 (.....), 4 (....), 2 (----), 1 (- - - -) and $\frac{1}{2}$ (......). In (a) and (b), $T\lambda_-$ and $T\lambda_+$ are plotted respectively.

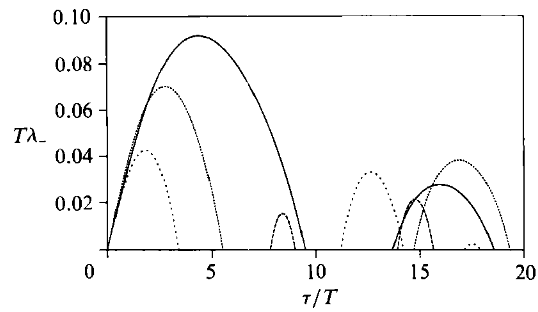

FIGURE 6. Graphs of the growth rates $T\lambda_-$ of magnetic field modes against τ/T for the ABC renewing flow with $h = 1$. We show results for various low-order rational values of k: $\frac{1}{3}$ (—), $\frac{1}{2}$ (....), $\frac{2}{3}$ (....) and 1 (----).

We are particularly interested in fast dynamo action for $\boldsymbol{k}$-vectors with rational components. For such wavevectors, the field and the flow are periodic with the same domain of periodicity on a sufficiently large scale. In figure 6 we plot $T\lambda_-$ as a function of τ/T for a few simple rational k-values. An interesting feature is that if $k < 1$, the dynamo operates for arbitrarily small τ/T, while $k = 1$ requires $\tau/T > 7$ for dynamo action. In other words, long time correlations are required for the amplification of fields possessing the same periodicity as the ABC renewing flow. The dynamo works best if it has room in which to stretch the field before twisting and folding it.

In order to confirm these results and verify that there is growth in typical realizations as well as in the ensemble average, we have undertaken some numerical simulations at zero magnetic diffusion. Consider the case with $a = q = h = \tau = T = 1$ and $k = \frac{1}{2}$; the magnetic field is periodic with periodicity box $V = [0, 4\pi]^3$. We take an initial magnetic field with negative helicity: $\boldsymbol{B}(\boldsymbol{x}, 0) = (\mathrm{i}, 1, 0)\exp(\frac{1}{2}\mathrm{i}z)$. In our numerical code we follow the evolution of field in a given realization of random waves and evaluate the projection onto the initial negative helicity mode,

$$P(n) = \tfrac{1}{2}(4\pi)^{-3}\int_V \mathrm{d}^3\boldsymbol{x}(-\mathrm{i}, 1, 0)\cdot\boldsymbol{B}(\boldsymbol{x}, n)\exp(-\tfrac{1}{2}\mathrm{i}z),$$

together with the energy density,

$$E_{\mathrm{M}}(n) = \tfrac{1}{2}(4\pi)^{-3}\int_V \mathrm{d}^3\boldsymbol{x}|\mathrm{Re}\,\boldsymbol{B}(\boldsymbol{x}, n)|^2.$$

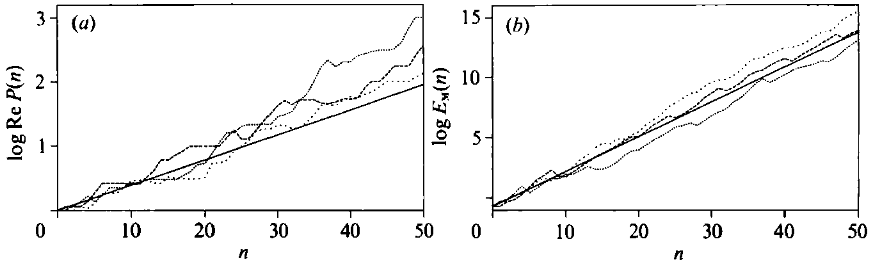

FIGURE 7. Growth of a magnetic field mode and magnetic energy in individual realizations. In (*a*) $\log \operatorname{Re} P(n)$ and in (*b*) $\log E_M(n)$ are plotted as functions of time $t = n$. The solid straight lines give the growth rates for the ensemble-averaged quantities. The other line styles correspond to different realizations.

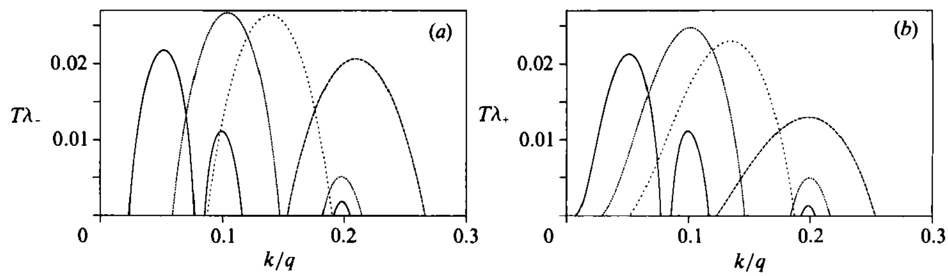

FIGURE 8. Graphs of the growth rates of magnetic field modes with positive and negative helicity against k/q for the isotropic renewing flow with zero average helicity and $\tau/T = 80$ (—), 40 (....), 30 (....) and 20 (----). In (*a*) and (*b*), $T\lambda_-$ and $T\lambda_+$ are plotted respectively.

If we ensemble average, $\langle P(n)\rangle = \sigma_-^n$ where $\sigma_- = (1+2J_0(\frac{1}{2})+\frac{1}{3}J_1(\frac{1}{2})) \approx 1.0397$ and $\langle E_M(n)\rangle = \frac{1}{2}(\frac{4}{3})^n$ (by a similar calculation to that in §3). The code evaluates the integrals by following the field from 5×10^6 randomly chosen points in V for each realization. The number of points was varied to check that each curve obtained is correct and the magnetic field structure well-resolved. In figure 7 we show the growth of $\operatorname{Re} P(n)$ and E_M(n) for several realizations. Both of these quantities grow at approximately the same rate as the ensemble averages (which are shown by solid straight lines); however the projection $\operatorname{Re} P(n)$ shows rather larger fluctuations than the energy. Because of phase-mixing, $\operatorname{Im} P(n)$ also grows (Hoyng 1987*a*, *b*); however, it remains small compared with $\operatorname{Re} P(n)$ during these simulations.

4.3. *Isotropic renewing flow with zero mean helicity*

In the two examples discussed above, the dynamo ceases to operate when $h = 0$. Now let us consider the effect of helicity fluctuations, so that instead of h taking a fixed value for all helical waves in our renewing flow, it is a random variable with a given probability distribution. If our ensemble of renewing flows is mirror-symmetric, the distribution for $-h$ is the same as that for h, and there is no dynamo action as $\langle hg_1(ak\tau, h)\rangle$ averages to zero. However, we can break mirror symmetry without the presence of average fluid helicity if we impose only the condition that $\langle h\rangle = 0$ but allow different distributions of positive and negative helicity fluctuations.

For a concrete example, let us take $h = 1$ with probability $\frac{1}{3}$ and $h = -\frac{1}{2}$ with probability $\frac{2}{3}$. Figure 8 shows growth rates of magnetic field modes against k/q for several large values of τ/T. The turnover time here is defined by (6), but with E_K

replaced by the average kinetic energy, $\langle E_K \rangle = \frac{3}{8}a^2$. Dynamo action is still possible in persistent flows with zero fluid helicity, provided that mirror symmetry is broken in some way. Gilbert, Frisch & Pouquet (1988) provide a deterministic example of a similar phenomenon.

5. Conclusions

We have studied the behaviour of magnetic fields in examples of renewing flows made up of random helical waves. The simple spatial and temporal structure of these flows enabled us to calculate analytically the growth of all single-point magnetic field moments in the absence of magnetic diffusion. In this case we find that a magnetic field becomes increasingly intermittent in time, as predicted by Zel'dovich *et al.* (1984). We discussed magnetic diffusion and studied normal modes of the mean magnetic field. When the fluid helicity is non-zero we see growth of magnetic field modes at zero diffusion, which, for renewing flows, implies fast dynamo action. If the flow is sufficiently persistent, having a long correlation time relative to the turnover time, magnetic fields with both signs of magnetic helicity may be amplified. We also found fast dynamo action in a renewing flow whose average fluid helicity is zero, but which lacks mirror symmetry.

There remain a number of problems to be explored. Our random flows are characterized by a fixed lengthscale, $1/q$, as are those of Drummond & Münch (1990). It would be interesting to construct renewing flows which incorporate a whole range of space and time scales (see Childress & Klapper 1991), in order to have a crude model of the effects of inertial-range turbulence. In a model with no preferred lengthscale, the results for magnetic field intermittency may be qualitatively different; according to Batchelor (1952) all the stretching exponents, γ_p should be equal in this case. Evidence supporting this hypothesis has been obtained by Malik (1990), who has measured the growth of surface areas (rather than lines) in random velocity fields made up of a range of scales.

For the case of non-zero magnetic diffusivity, we limited ourselves to finding the evolution of the mean magnetic field. This demonstrated that fast dynamo action occurs, but gives us no information about the distribution of the quadratic quantities, such as energy and helicity, in magnetic fluctuations of different scales. This information is contained in the 2-point second moment of the field. It would be worth studying the evolution of the second moment in the renewing flows introduced in this paper. This would extend the work of Kazantsev (1967) and Novikov *et al.* (1983) to flows with finite correlation times and non-zero helicity. Such a study may lead to insight into the transfer and dissipative processes occurring in the fast dynamos that occur in these renewing flows. Finally we have seen that growth of magnetic field modes occurs in typical realizations of random flows; however, it would be interesting to carry out further numerical simulations of random dynamos to understand stretching and folding mechanisms, and study the field structure in individual realizations.

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